ELEMENTS OF THE
ELECTROMAGNETIC THEORY OF LIGHT

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BY

LUDWIK SILBERSTEIN, Ph.D.
LECTURER IN NATURAL PHILOSOPHY AT THE UNIVERSITY OF ROME

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PREFACE.

THIS little volume, whose object is to present the essentials of the electromagnetic theory of light, was rewritten, at the instance of Messrs. Adam Hilger, Limited, from my Polish treatise on Electricity and Magnetism (3 vols., Warsaw, 1908-1913, published by the kind help of the Mianowski Institution). It consists principally of an English version of chapter viii., vol. ii., of that work with some slight omissions and modifications. In order to make the subject accessible to a larger circle of readers Section 3 was added. The language adopted is mainly vectorial. This is the chief reason of the compactness of the book which, it is hoped, notwithstanding its small number of pages, will be found to contain an easy and complete presentation of the fundamental part of Maxwell's theory of light.

I gladly take the opportunity of expressing my best thanks to Messrs. Hilger for enabling me to submit a portion of my treatise to the English reader.

L. S.

LONDON, *May*, 1918.

CONTENTS.

1. The Origin of the Electromagnetic Theory.

The electromagnetic theory of light, now for many years in universal acceptance, was proposed and developed by James Clerk Maxwell about the year 1865.* By elimination, from his classical differential equations, of the electric current Maxwell has obtained, for the "vector potential" $\mathfrak{A}$,† a differential equation of the second order which in the case of a non-conducting isotropic medium has assumed the form

$$K\mu\frac{\partial^2\mathfrak{A}}{\partial t^2} = \nabla^2\mathfrak{A} \quad . \quad . \quad . \quad . \quad [M]$$

where ∇^2 is the Laplacian (Maxwell's $-\nabla^2$, borrowed from Hamilton's calculus of quaternions). Maxwell's coefficients, the "specific inductive capacity" K, and the magnetic "permeability" μ, are not pure numbers. Let c be the ratio of the electromagnetic unit of electric charge to the electrostatic unit of charge. Then Maxwell's coefficients are such that, for air (or vacuum),

$$K = 1,\ \mu = \frac{1}{c^2},\ \text{in the electrostatic system,}$$

$$K = \frac{1}{c^2},\ \mu = 1,\ \text{in the electromagnetic system.}$$

* *Phil. Trans.*, 1865, p. 459 *et seq.*, reprinted in *Scientific Papers.* See also *Treatise on Electricity and Magnetism*, vol. ii., chap. xx.

† Which, in absence of a purely electrostatic potential, gives the electric force by its negative time derivative, *i.e.* in the notation to be adopted throughout this volume, $\mathbf{E} = -\partial\mathfrak{A}/\partial t$.

Thus, in either system, $K\mu = 1/c^2$, for air. Now, from his above equation which in the case of plane waves, for instance, reduces to

$$K\mu \frac{\partial^2 \mathfrak{A}}{\partial t^2} = \frac{\partial^2 \mathfrak{A}}{\partial x^2},$$

Maxwell concluded at once that the velocity of propagation of electromagnetic disturbances should be

$$v = \frac{1}{\sqrt{K\mu}}$$

in any medium, and therefore, in air, $v = c$.

Thus Maxwell has arrived at the capital conclusion that "*the velocity of propagation in air* [*or in vacuo*] *is numerically equal to the number of electrostatic units contained in an electromagnetic unit of electric charge*". The dimensions of this "number" c, or ratio of units, are obviously those of a *velocity*. For, by what has just been said, we have the dimensional equation

$$[c^2 t^2] = [x^2]$$

where x is a length and t a time.

Now, the experimental measurements of Kohlrausch and Weber,* famous in those times, have given for the ratio of the two units of charge the value

$$c = 310740 \text{ km. sec.}^{-1} = 3{\cdot}107 \,.\, 10^{10} \text{ cm. sec.}^{-1},$$

or rather, after account has been taken of W. Voigt's corrections (*Ann. d. Phys.*, vol. ii.), $3{\cdot}111 \,.\, 10^{10}$ cm. sec.$^{-1}$. Maxwell quotes also the value obtained from a comparison of the units of electromotive force † by William Thomson (1860),

* Kohlrausch and Weber, *Elektrodyn. Maassbestimmungen, etc.*; W. Weber, *Elektrodyn. Maassbestimmungen, insbesondere Widerstandmessungen.*

† An electromagnetic unit of electromotive force contains $1/c$ electrostatic units.

$$c = 2{\cdot}82 \,.\, 10^{10} \text{ cm. sec.}^{-1},$$

and the value which he has later (1868) obtained himself from a comparison of the same units,

$$c = 2{\cdot}88 \,.\, 10^{10} \text{ cm. sec.}^{-1}.$$

These figures Maxwell has compared with those obtained for *the velocity of light* in air and in interstellar space:—

$3{\cdot}14 \,.\, 10^{10}$ cm. sec.$^{-1}$	.	(Fizeau).
$3{\cdot}08$,, ,, ,,	.	(astronom. observations).
$2{\cdot}98$,, ,, ,,	.	(Foucault).

The agreement of the light velocity with that ratio of units c has thus turned out to be satisfactory. And, although Maxwell himself states it very cautiously by saying only that his theory is not contradicted by these results, there can be but little doubt that the said agreement has had a decisive influence upon the birth of the electromagnetic theory of light. And later measurements of both the velocity of light and the ratio of units have by no means shattered the belief in the agreement and even the identity of these two magnitudes which, to judge from their original physical meaning, would seem to have hardly anything in common with one another.*

For a transparent isotropic medium differing from air through its dielectric "constant" K, and showing but a negligible difference in μ, Maxwell's theory gave the velocity of light $c/\sqrt{K}$, where K is taken in the electrostatic system and is thus a pure number. The refractive index of the dielectric medium with respect to air should therefore be given by

$$n = \sqrt{K}.$$

* Many numerical data, together with the bibliography of the subject up to 1907, will be found in *Encyklop. d. mathem., Wiss.*, vol. v., part 3, p. 186 *et seq.*; Leipsig, 1909. Interesting details will be found in Prof. Whittaker's precious *History of the Theories of Aether and Electricity*, chapter viii., Longmans, Green & Co., 1910.

In order to test this predicted relation, Maxwell quotes the only example of paraffin. For solid paraffin Gibson and Barclay have found

$$K = 1{\cdot}975.$$

On the other hand Maxwell takes the values of the refractive index n found by Gladstone for liquid paraffin (at 54° and 57° C.) for the spectrum lines A, D, and H, from which he finds, by extrapolation, *for infinitely long waves*

$$n = 1{\cdot}422.$$

He takes infinitely long waves in order to approach as well as possible the conditions of the slow processes (static or quasi-static) upon which the measurements of the dielectric coefficient K were based. Putting together the values thus obtained,

$$n = 1{\cdot}422$$

$$\sqrt{K} = 1{\cdot}405.$$

Maxwell confesses that their difference is too great to be thrown on the experimental errors; he does not doubt, however, that if $\sqrt{K}$ is not simply equal to the refractive index, yet it makes up its essential part. He expects a better agreement only when the grain structure of the medium in question will be taken into account.

It is universally known that Maxwell's predictions have found a splendid corroboration a quarter of a century later,* in the famous experiments of Hertz who has not only confirmed the existence of electromagnetic waves, but also verified the approximate equality of their velocity of propagation with

* In 1889. Hertz's papers are reprinted in vol. ii. of his *Gesammelte Werke*, under the title, *Untersuchungen über die Ausbreitung der elektrischen Kraft*; Leipzig, 1892. English version in *Miscellaneous Papers*, translated by Jones and Schott.

that of light by measuring the length of stationary waves and by calculating, on the other hand, the period of his electric oscillator by the well-known approximate formula $T = \frac{2\pi}{c}\sqrt{LC}$. The agreement was satisfactory; a better one could, at any rate, not be expected, seeing that the self-induction and the capacity of the oscillator (L, C) entering into the above formula corresponded to quasi-stationary conditions while Hertz's oscillations were of a rather high frequency. Moreover, it is well known that Hertz and his numerous followers have imitated, with short electromagnetic waves, almost all the fundamental optical experiments.

2. Advantages of the Electromagnetic over the Elastic Theory of Light.

It will be well to acquaint the reader with certain conspicuous advantages offered by the electromagnetic as compared with the "elastic" theory of light, *i.e.* the theory based upon the assumption of an elastically deformable æther. In doing so we shall by no means attempt to give here the complete history of the luminiferous æther, but shall content ourselves with sketching a certain fragment of that complicated and interesting history, *viz.* that concerning the question of *longitudinal waves* (which had at any cost to be got rid of) and of the so-called *boundary conditions*. With this aim in view it will be enough to start from Green's work leaving aside the earlier investigations of Fresnel, F. Neumann, and others.

Green's * æther is a continuous elastic medium endowed

* G. Green, *On the Laws of Reflexion and Refraction of Light at the Common Surface of two Non-crystallised Media*, Cambridge Phil. Trans., 1838, reprinted in *Mathematical Papers*, pp. 245-69. (In this paper Green denotes our following n by B and $k + \frac{4}{3}n$ by A.) *On the Propaga-*

with a certain resistance against compression k, *i.e.* with a *compressibility* $1/k$, with an elasticity of shape or "*rigidity*" n, and, finally, with a *density*, or inertia for unit volume, ρ. In isotropic bodies or media k, ρ, n are ordinary scalars, while in crystals n has in different directions different values or is, in modern terminology, a linear vector operator; k, ρ retain their scalar character. In such an æther, two kinds of waves can exist and propagate themselves independently of one another, *viz. longitudinal* or dilatational waves, with the velocity

$$v' = \sqrt{\frac{k + 4n/3}{\rho}} \quad . \quad . \quad . \quad . \quad (1)$$

and *transversal* or distortional ones, with the velocity

$$v = \sqrt{\frac{n}{\rho}} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in the case of isotropy. Owing, however, to the discovery of the phenomena of polarization by Malus, Arago, and Fresnel, the physicists have convinced themselves that the luminous oscillations must be purely transversal. In order, therefore, to harmonize the theory with experience it has been necessary to get rid of the longitudinal waves. At first sight it would seem that it is enough to simply assume that only the transversal oscillations of the æther, and not the longitudinal ones, affect the retina. This, however, would not solve the difficulty. In fact, it can easily be shown that in the process of refraction at the boundary of two optically different media, purely transversal vibrations of the æther in the first medium,* penetrating into the second, would split into two wave-trains: a transversal and a longitudinal one.

tion of Light in Crystallized Media, Cambridge Phil. Trans., 1839, reprinted in *Math. Pap.*, pp. 293-311.

* To wit, those contained *in the plane of incidence.*

The velocities of propagation of these refracted waves being different, the corresponding two rays would make with one another, in general, a non-evanescent angle; each of these rays on emerging from the second into the first medium, through any other boundary, would again be split into two: one consisting of transversal and another of longitudinal vibrations, and the *two* rays of *transversal* vibrations thus originated would then certainly be accessible to our senses. In short, we should have a peculiar phenomenon of double refraction of light in an isotropic body. No traces, however, of such a phenomenon have ever been found experimentally. The longitudinal waves, therefore, had to be got rid of in a more radical way.

Now, the mathematical investigation of the subject has shown that the superfluous dilatational wave in the second medium dies away almost completely within a few wave-lengths from the boundary surface if it is assumed either that

(1) The velocity of propagation v' of the longitudinal vibrations is *very large* as compared with the velocity v of the transversal ones, or—

(2) That this ratio, $v' : v$, of the velocities is *very small*.

As will be seen later on, William Thomson has established (1888) the physical admissibility of the second assumption. Green, however, was convinced that the first was the only possible assumption, since he did not see his way to admit a negative k, *i.e.* a negative compressibility: a body endowed with a negative k would, at the slightest difference between its own pressure and that of its surrounding medium, expand or shrink indefinitely. For Green, therefore, the lower limit of k has been $k = 0$, and consequently, the lower limit of the ratio of the two velocities, by [1] and [2],

$$\frac{v'}{v} = \frac{2}{\sqrt{3}}.$$

He has thus been compelled to adopt the first of the two

assumptions, *viz.* that *the dilatational waves are propagated with a velocity which is enormously greater than that of the transversal ones*, in other words, that the ratio of the coefficients *k/n is very large.* In this manner Green's æther has become similar to an almost incompressible jelly.

Next, in order to obtain from the differential equations of motion of his æther and from the boundary conditions, the laws of reflection and refraction at the interface of two isotropic media 1 and 2, Green introduces the supplementary assumption that the rigidity of the æther in the two media is the same, while its density has different values,

$$n_1 = n_2, \ \rho_1 \neq \rho_2.$$

This gives for the intensities of the reflected and the refracted ray, in the case of incident vibrations *normal to the plane of incidence*, two formulæ identical with Fresnel's formulæ for light polarized in the plane of incidence which are notoriously in good agreement with the experimental facts. Thus far, however, Green makes no use of his assumption k/n equal to a large number; for in the case in question the longitudinal waves do not enter into play. They reassert themselves only when the incident light oscillates *in the plane of incidence.* Now, the formulæ which in the latter case follow from Green's theory, do not agree with the corresponding Fresnelian formulæ and deviate very sensibly from experiment; in fact, they give for light reflected under the "angle of polarization" an intensity which differs too much from zero.* This is a serious objection against Green's theory.

The substitution, for Green's $n_1 = n_2$, $\rho_1 \neq \rho_2$, of the opposite assumption of Neumann or of MacCullagh:

* Fresnel's formula gives *zero* for that intensity, while actually but a certain minimum is observed under the "angle of polarization"; this minimum, however, although still observable, is very weak as compared with the intensity of the incident light, and is most likely due to a heterogeneous transition layer at the interface of the two media.

$$\rho_1 = \rho_2, \ n_1 \neq n_2$$

with the retention of all the remaining points of Green's theory, does not help the matter. In fact, the investigations of W. Lorenz (1861) and of Lord Rayleigh (1871), based upon the last assumption, lead to a result which emphatically contradicts experience, *viz*. to a formula for the ratio of amplitudes of reflected and incident light which, in the case of but slightly differing refractive indices of the two media, can be written

$$A_r : A_i = \text{const. } \sec^2 i \, . \cos 4i, *$$

where i is the angle of incidence. This ratio vanishes for $i = \pi/8$ and for $i = 3\pi/8$; thus we should have *two different* angles of polarization—a phenomenon which nobody has ever observed.

It has been necessary, therefore, to return once more to Green's assumption $n_1 = n_2, \rho_1 \neq \rho_2$, and to meet the reflection and refraction difficulties by modifying Green's theory in some other direction. This has been done by Sir William Thomson who has replaced Green's jelly by a kind of *foam* æther which will be described presently.

Thus far we have been concerned with isotropic media. In *anisotropic* media, *viz*. in optically biaxial crystals, Green's æther † had three principal rigidities, n_1, n_2, n_3, a single scalar coefficient k (as in isotropic media), and a constant density ρ, the coefficient k being again *very large* as compared with each of the three rigidities. We know already that under such circumstances the velocity (v') of longitudinal waves is very great as compared with the velocity (v) of transversal ones. For the latter, Green's theory gives at once the universally known *Fresnelian equation*

* W. Lorenz, *Pogg. Ann.*, vol. cxiv., 1861, pp. 238-50; Lord Rayleigh, *Phil. Mag.* for August, 1871, see especially p. 93.

† *On the Propagation of Light in Crystallised Media*, already quoted, p. 200.

$$\frac{l_1^2}{v^2 - n_1/\rho} + \frac{l_2^2}{v^2 - n_2/\rho} + \frac{l_3^2}{v^2 - n_3/\rho} = 0, \qquad [3]$$

a result which is in excellent agreement with experimental facts. (In this formula, l_1, l_2, l_3, are the direction cosines of the wave normal with respect to the principal rigidity axes.) Thus far the propagation of light in a crystalline medium. Difficulties, however, arise in connexion with the treatment of reflection and refraction at the surface of a crystal, in contact with, say, an isotropic medium. For optically uniaxial crystals ($n_2 = n_3$, $n_1 \neq n_2$) one could, after all, accept Green's assumption according to which *the* principal rigidity of the crystal (n_1) corresponding to its unique rigidity axis should be equal to the rigidity of the æther in the adjacent isotropic medium. In the case, however, of optically *biaxial* crystals, having three different rigidities n_1, n_2, n_3, one could hardly privilege any one of them, *i.e.* put it equal to the æther rigidity in the adjacent medium. And if we wished to meet this difficulty by assuming that the *densities* of the æther in the crystal and the adjacent medium are equal to one another, the previous, undesirable result would reappear, *viz.* two different angles of polarization (as in the case of ordinary, non-crystalline reflection).

Lord Rayleigh * attempts to improve this weak point of Green's theory by assuming that the æther within biaxial crystals moves so as if it had in three orthogonal directions *three different principal densities*, ρ_1, ρ_2, ρ_3, and ordinary scalar elastic coefficients k, n, independent of direction and equal for all media. In passing from one medium to another it is the æthereal density only which is changed. This theory, involving a peculiar dependence of the æther's inertia upon the direction of motion, is based upon reasonings concerning the mutual action of the æther and the molecules of ponderable matter. And it is precisely that interaction which is supposed to be the source of those directional properties of

* *Phil. Mag.* for June, 1871.

the æther's mass. Now, Rayleigh's differential equations give for the square of the velocity of propagation of all possible waves (without, thus far, the exclusion of the longitudinal ones) the cubic equation *

$$\frac{l_1^2}{V^2 - B/\rho_1} + \frac{l_2^2}{V^2 - B/\rho_2} + \frac{l_3^2}{V^2 - B/\rho_3} = \frac{A}{(A - B)V^2} \qquad [4]$$

where $B = n$, $A = k + 4n/3$. Introducing here again, after Rayleigh, Green's original assumption $A/B = \infty$, we have

$$\frac{l_1^2/\rho_1}{V^2(V^2 - B/\rho_1)} + \frac{l_2^2/\rho_2}{V^2(V^2 - B/\rho_2)} + \frac{l_3^2/\rho_3}{V^2(V^2 - B/\rho_3)} = 0,$$

that is to say, for one of the three waves, *viz.* the longitudinal, $V = v' = \infty$, as in Green's theory, and for the remaining two, transversal ones, the cubic equation

$$\frac{l_1^2/\rho_1}{v^2 - B/\rho_1} + \frac{l_2^2/\rho_2}{v^2 - B/\rho_2} + \frac{l_3^2/\rho_3}{v^2 - B/\rho_3} = 0.$$

But this equation does not agree with Fresnel's equation [3] which is notoriously a faithful representation of the experimental facts. Thus, Rayleigh's theory, in its turn, had to undergo further radical modifications.

Let us return for another moment to the more general equation [4]. We see at a glance that it will yield, with any degree of approximation, the required Fresnelian equation, provided it is assumed that the ratio $A : B$ $\left(\text{or } \frac{k}{n} + \frac{4}{3}\right)$, instead of being very great is, on the contrary, *very small*, that is to say, if we decide in favour of the second of the above alternatives which was rejected by Green *a priori*.

It is precisely this assumption (2), page 7, *i.e.*

$$A : B = \frac{k}{n} + \frac{4}{3} = \textit{a very small number},$$

* Glazebrook, *Phil. Mag.* (5), vol. xxvi., p. 521.

which is the starting-point of Thomson's æther theory (1888).* It is true that previous authors have already used that assumption. Fresnel has ascribed to his æther now an evanescent and now an infinite compressibility. Thomson, however, was the first to prove the physical possibility of that assumption. In fact (2) requires at any rate $A/B < \frac{4}{3}$, and therefore $k < 0$, that is to say, a *negative* compressibility, and that was the reason why Green thought that the æther's stability calls for $A/B > \frac{4}{3}$. Thomson proves, however, that this is by no means a necessary condition of the stability of the æther.

To see this, let us assume, after Thomson, that the æther either occupies a limited region of space and is *fixed* at its boundary, or extends indefinitely in all directions but that the displacements, ξ, etc., of its particles decrease in such a manner that the products $\xi\partial\xi/\partial x$, etc., become infinitesimals at least of *the third order.* Then the expression for the work to be done upon the æther in deforming it infinitesimally from its natural (or neutral) state can, by partial integration, be reduced to the form

$$W = \int[\tfrac{1}{2}A\sigma^2 + 2B\omega^2]d\tau \quad . \quad . \quad . \quad [5]$$

where σ is the dilatation, or the *div* of the displacement, and ω the elementary rotation, or the $\frac{1}{2}$ *curl* of the displacement, the integral being extended over the whole volume (τ) of the æther. Now, this work will be positive, and therefore the neutral state of the æther a state of stable equilibrium, *provided that A and B are positive.* The lower limit of each of these coefficients is thus *zero*, so that the ratio $A : B$, *i.e.* the ratio of the velocities of longitudinal and of transversal waves

* Sir William Thomson, *On the Reflexion and Refraction of Light*, *Phil. Mag.*, vol. xxvi. (November, 1888), pp. 411-25.

can be made as small as we like. If the free æther is attributed a certain density ρ, its rigidity must be made so great as to give, in round figures, $\sqrt{B/\rho} = 3 \cdot 10^{10}$ cm. sec.$^{-1}$, and (in Thomson's theory) A very small as compared with that rigidity, so that k, the inverse of compressibility, is negative and its absolute value but insensibly smaller than $\frac{4}{3}B$. With regard to stability, an æther thus conceived would resemble a mass of foam filling out the interior of a rigid shell or closed vessel and once and for ever attached to its surface. Owing to this analogy, pointed out by Thomson himself, his æther is known as the *foam-æther*. In application to the reflection and refraction of light at the boundary of two isotropic media it gave at once the two groups of *Fresnelian formulæ*, for vibrations contained in and perpendicular to the plane of incidence. In deriving these formulæ Thomson assumes that the rigidity B has in both media the same value which, owing to the structure of the said æther, is also a direct consequence of the continuity of the tangential component of the pressure across the interface.

The conception of the foam-æther, combined with Rayleigh's theory, mentioned a moment ago, *i.e.* with formula [4] for small $A : B$, leads to a system of crystalline optics agreeing with experiment, *viz.* giving not only Fresnel's equation for the velocity of propagation within crystals but also reflection and refraction formulæ identical with the corresponding formulæ of the electromagnetic theory.

Of all the mechanical or "elastic" theories of light hitherto invented, Thomson-Rayleigh's theory can, at any rate, claim to be the best.

The purpose, however, of bringing before the reader this brief historical sketch has been of another kind, *viz.* to help the reader to perceive the superiority of the electromagnetic theory over the elastic theories of light. We have seen, in fact, what desperate struggles the elastic theory had to sustain against the intrusive and obstinate longitudinal waves,

and how many difficulties it had to overcome in connexion with the boundary conditions and with crystalline optics. Various hypotheses as to the particular properties of the æther in homogeneous media and as to its behaviour in optically different bodies had to be tried and rejected. And that variety of concepts would be even more striking if our historical sketch were fuller. Against this the electromagnetic theory of light, based upon Maxwell's differential equations, has the great methodological advantage that it requires, neither for the abolition of the longitudinal waves nor for the establishment of the boundary conditions, any supplementary hypotheses. In fact, the waves deduced from Maxwell's equations are *purely transversal*, that is to say, both the oscillating vectors, **E** and **M**, are, in an isotropic medium, normal to the direction of propagation.* And the boundary conditions for each of these vectors follow directly from these equations in a perfectly definite way, provided that the ratios of the coefficients K and μ for the two adjacent media, 1 and 2, are known. The elastic theory looked for the origin of the optical difference of the two bodies either in the difference of the densities, $\rho_1 \neq \rho_2$, or in that of the rigidities, $n_1 \neq n_2$, of the æther contained within these bodies. The electromagnetic theory needs in this respect no hypotheses. It is known from experience that at least all the transparent bodies have very nearly *equal magnetic permeabilities* ($\mu \doteq 1$), namely, the same as the vacuum, while they *differ greatly with regard to their dielectric coefficients or permittivities* K. It is therefore perfectly natural that the electromagnetic theory throws the optical difference of bodies upon the permittivity K, and not upon the permeability μ. It is true that the values of K, in its primary meaning, are known experimentally only from quasi-static processes, or such as imply electromagnetic waves considerably longer than the proper light waves. But, at any rate, the funda-

* *Vide infra*, p. 20.

mental, qualitative choice of K, and not of μ, as the coefficient decisive for the optical behaviour of different bodies, is not built upon a hypothesis but is based upon experience.

It is well known that for a small number of bodies the permeability μ, derived from quasi-static or slow processes, acquires considerable, and sometimes even enormous values. Experience has taught us, however, that in considering optical phenomena, *i.e.* very short electromagnetic waves, we have to put

$$\mu = 1$$

for all bodies, without the exception even of cobalt, nickel, and iron. In other words, in presence of rapid luminous vibrations, all bodies behave with respect to their magnetic permeability as empty space itself.

While the elastic theory has ascribed the optical anisotropy of *crystals* at one time to the æther's rigidity, and at another to its density, the electromagnetic theory attributes it entirely to the dielectric coefficient, putting again $\mu = 1$ for all directions, and treating K, in full harmony with experience, as *a linear vector operator*. In optically uniaxial crystals this operator is axially symmetric, and in optically biaxial crystals it has three mutually perpendicular principal axes to which correspond three different principal permittivities K_1, K_2, K_3.

Fresnel's equation for crystals, as was shown by Maxwell himself, follows rigorously from the electromagnetic theory without any auxiliary hypotheses. And with the help of the vectorial method Maxwell's simple proof can be still further abbreviated.

For ordinary reflection and refraction of light the electromagnetic theory gives at once both groups of Fresnel's formulæ. Also the results deduced from the electromagnetic theory for crystalline refraction are in harmony with experimental facts.

Maxwell's monumental work, his original electromagnetic

theory of light, calls for certain modifications or additions only when one comes to face the problems of the optics of moving bodies, of dispersion and absorption and of magneto- and electro-optics. The process of light emission itself lies, of course, outside the frame of Maxwell's original theory. These more refined, and at the same time less solidly built parts of optics will not be treated in the present little volume, its purpose being to acquaint the reader only with the chief points of the subject.

3. Maxwell's Equations. Plane Waves.

The reader is supposed to have a certain knowledge of the elements of the theory of electricity and magnetism. Here, therefore, it will be enough to recall the fundamental differential equations upon which the whole of our subject is based. Maxwell's equations, in Hertz-Heaviside's form, for a transparent (isolating), homogeneous medium, are, in vector language,

$$\left.\begin{aligned} K\frac{\partial \mathbf{E}}{\partial t} = c.\,\mathrm{curl}\,\mathbf{M}\,; \quad \frac{\partial \mathbf{M}}{\partial t} = -\,c.\,\mathrm{curl}\,\mathbf{E}\,; \\ \mathrm{div}\,(K\mathbf{E}) = \mathrm{div}\,\mathbf{M} = 0\,, \end{aligned}\right\} \tag{1}$$

where the vectors **E**, **M**, stand for the electric and the magnetic forces, and $c = 3{\cdot}10^{10}$ cm. sec.$^{-1}$ is the light velocity in empty space. For isotropic media, K is an ordinary scalar, the permittivity, and in the most general case of a crystalline medium, K is a (self-conjugate) linear vector operator which can be called the *permittivity* operator. As to the magnetic permeability, we have, in accordance with what has been said in the preceding section, assumed it to be equal to unity, as for empty space. In Cartesians, E_1, etc., M_1, etc., being the components of the vectors along the axes of x, y, z,* coinciding with the principal axes of K, the above equations are

* Right-handed system.

$$\left.\begin{array}{l|l}
\dfrac{K_1}{c}\dfrac{\partial E_1}{\partial t}=\dfrac{\partial M_3}{\partial y}-\dfrac{\partial M_2}{\partial z} & \dfrac{1}{c}\dfrac{\partial M_1}{\partial t}=\dfrac{\partial E_2}{\partial z}-\dfrac{\partial E_3}{\partial y} \\
\dfrac{K_2}{c}\dfrac{\partial E_2}{\partial t}=\dfrac{\partial M_1}{\partial z}-\dfrac{\partial M_3}{\partial x} & \dfrac{1}{c}\dfrac{\partial M_2}{\partial t}=\dfrac{\partial E_3}{\partial x}-\dfrac{\partial E_1}{\partial z} \\
\dfrac{K_3}{c}\dfrac{\partial E_3}{\partial t}=\dfrac{\partial M_2}{\partial x}-\dfrac{\partial M_1}{\partial y} & \dfrac{1}{c}\dfrac{\partial M_3}{\partial t}=\dfrac{\partial E_1}{\partial y}-\dfrac{\partial E_2}{\partial x} \\
\multicolumn{2}{c}{K_1\dfrac{\partial E_1}{\partial x}+K_2\dfrac{\partial E_2}{\partial y}+K_3\dfrac{\partial E_3}{\partial z}=0} \\
\multicolumn{2}{c}{\dfrac{\partial M_1}{\partial x}+\dfrac{\partial M_2}{\partial y}+\dfrac{\partial M_3}{\partial z}=0.}
\end{array}\right\} \quad (1')$$

In the case of isotropy $K_1 = K_2 = K_3 =$ an ordinary scalar K. The last two equations will be shortly referred to as the solenoidal conditions. A glance upon (1′) suffices to show the waste of paper (and of time) involved in the Cartesian expansion of formulæ, (1) in the present case, which are, by their intrinsic nature, vectorial.*

As to the immediate general consequences of (1), it will be enough to mention those concerning the electromagnetic energy and its flux. The density of energy is, in the general case of a crystalline medium, given by

$$u = \tfrac{1}{2}(\mathbf{E}K\mathbf{E} + M^2) \quad . \quad . \quad . \quad (2)$$

Now multiplying (scalarly) the first of (1) by $\mathbf{E}$, the second by $\mathbf{M}$, and adding both, we have

$$\frac{1}{c}\frac{du}{dt} = \mathbf{E}\,\mathrm{curl}\,\mathbf{M} - \mathbf{M}\,.\,\mathrm{curl}\,\mathbf{E} = -\,\mathrm{div}\,\mathrm{V}\mathbf{EM}.$$

Thus the *flux of energy*, per unit time and unit area, is seen to be

$$\mathbf{F} = c\,\mathrm{V}\mathbf{EM} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

* The Cartesian splitting will be avoided as much as possible. Those readers who are not familiar with vectors can acquire whatever is necessary to follow the present deductions by reading chapter i. of the author's *Vectorial Mechanics*, Macmillan, 1913.

This vector, c times the vector product of **E** and **M**, is universally known as the *Poynting vector*. It is this vector which, by its direction, defines the optical *ray*, in an isotropic medium as well as in a crystal. If **E** points upward and **M** to the right, the flux, and, therefore, the ray is directed forward. The *intensity* of light at a given point is measured by the time average of the density u of electromagnetic energy.

As a preparation for the following sections we need only those integrals of the equations (1) which correspond to *plane waves*. Let the unit vector **n** be the *wave-normal*, and let the scalar distance s be measured along **n**. Then, by the very definition of plane waves, the vectors **E**, **M**, depend only on s and the time t. Under these circumstances the Hamiltonian ∇ becomes

$$\nabla = \mathbf{n}\frac{\partial}{\partial s},$$

and, therefore, for any vector **R**,

$$\operatorname{curl}\mathbf{R} = \mathrm{V}\nabla\mathbf{R} = \mathrm{V}\mathbf{n}\frac{\partial\mathbf{R}}{\partial s},$$

and

$$\operatorname{div}\mathbf{R} = \nabla\mathbf{R} = \mathbf{n}\frac{\partial\mathbf{R}}{\partial s}.$$

Thus, for any plane waves, equations (1) become

$$\frac{K}{c}\frac{\partial\mathbf{E}}{\partial t} = \mathrm{V}\mathbf{n}\frac{\partial\mathbf{M}}{\partial s};\ \frac{1}{c}\frac{\partial\mathbf{M}}{\partial t} = -\mathrm{V}\mathbf{n}\frac{\partial\mathbf{E}}{\partial s},$$

$$\mathbf{n}\frac{\partial K\mathbf{E}}{\partial s} = \mathbf{n}\frac{\partial\mathbf{M}}{\partial s} = 0,$$

or, since **n** is constant in space,

$$\frac{K}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{\partial}{\partial s}\,\mathrm{V}\mathbf{n}\mathbf{M}\,;\ \frac{1}{c}\frac{\partial \mathbf{M}}{\partial t} = \frac{\partial}{\partial s}\,\mathrm{V}\mathbf{E}\mathbf{n}, \tag{4a}$$

$$\frac{\partial}{\partial s}(K\mathbf{E}\,.\,\mathbf{n}) = \frac{\partial}{\partial s}(\mathbf{M}\mathbf{n}) = 0 \tag{4b}$$

In what follows we shall be concerned with monochromatic light, *i.e.* with simple periodic waves. Then $\mathbf{E}$, $\mathbf{M}$ are proportional to $e^{im(s-vt)}$, where $i = \sqrt{-1}$, m = const. ($= 2\pi$ divided by the wave-length in the given medium), and v the velocity of propagation, so that $\frac{c}{v}$ is the refractive index of the medium. Thus

$$\frac{\partial}{\partial t} = -imv,\ \frac{\partial}{\partial s} = im,$$

and the equations (4a) become, independently of m,

$$\frac{v}{c}K\mathbf{E} = \mathrm{V}\mathbf{M}\mathbf{n}\,;\quad \frac{v}{c}\mathbf{M} = \mathrm{V}\mathbf{n}\mathbf{E}.$$

The solenoidal conditions (4b), which now require $K\mathbf{E}\mathbf{n} = \mathbf{M}\mathbf{n} = 0$, are already satisfied,* since $\mathbf{n}\mathrm{V}\mathbf{M}\mathbf{n} = \mathbf{n}\mathrm{V}\mathbf{n}\mathbf{E} = 0$, identically.

Let $\mathbf{D}$ be the *dielectric displacement*, *i.e.*

$$\mathbf{D} = K\mathbf{E}.$$

Then the last pair of equations will become

$$\frac{v}{c}\mathbf{D} = \mathrm{V}\mathbf{M}\mathbf{n};\ \frac{v}{c}\mathbf{M} = \mathrm{V}\mathbf{n}\mathbf{E} \quad . \quad . \quad . \tag{5}$$

These will be our fundamental equations, valid for plane waves of wave-normal $\mathbf{n}$, in any homogeneous medium, be it

* The trivial case $v = 0$ being, of course, disregarded.

isotropic or crystalline. They contain, as a consequence, the solenoidal conditions

$$\mathbf{Dn} = 0, \quad \mathbf{Mn} = 0,$$

whose plain meaning is: the magnetic force and the dielectric displacement are *perpendicular to the wave-normal*, *i.e.* are contained in the plane of the wave. In other words, $\mathbf{M}$ and $\mathbf{D}$ (not $\mathbf{E}$, in general) are purely transversal.

Again, multiplying the first of (5) by $\mathbf{M}$, and the second by $\mathbf{E}$, we have

$$\mathbf{MD} = \mathbf{EM} = 0,$$

i.e. $\mathbf{M} \perp \mathbf{E}, \mathbf{D}$. But, in general, $\mathbf{En} \neq 0$.

We shall return to the general relations (5) when we come to treat, in detail, the optical properties of crystals.

For the present let us confine ourselves to *isotropic* media. Then $\mathbf{D}$ coincides with $\mathbf{E}$ in direction, being simply K times $\mathbf{E}$. Thus, in isotropic media, we have not only $\mathbf{E} \perp \mathbf{M}$ and $\mathbf{M} \perp \mathbf{n}$, but also $\mathbf{E} \perp \mathbf{M}$. In short, $\mathbf{E}$, $\mathbf{M}$, $\mathbf{n}$, are all mutually perpendicular. Again, multiplying, say, the second of (5) by $\mathbf{M}$, we have

$$\frac{v}{c} M^2 = \mathbf{MVnE} = \mathbf{nVEM} \quad . \quad . \quad . \quad (6)$$

or also, by (3),

$$vM^2 = \mathbf{nF}. \quad . \quad . \quad . \quad . \quad (6a)$$

From (6) we see that $\mathbf{nVEM}$ is always positive, *i.e.* that

$$\mathbf{E}, \mathbf{M}, \mathbf{n}$$

is a *right-handed* system of orthogonal vectors.

And since $\mathbf{F}$ is concurrent with $\mathbf{n}$,

$$\mathbf{E}, \mathbf{M}, \mathbf{ray}$$

Further, multiplying the first of (5) by $\mathbf{E}$, and remembering that $\mathbf{D} = K\mathbf{E}$, we have

$$\frac{v}{c}KE^2 = \mathbf{n}\mathrm{V}\mathbf{EM} = \frac{v}{c}M^2.$$

Thus $$KE^2 = M^2, \quad . \quad . \quad . \quad . \quad (7)$$

i.e. half of the energy is electric, and half magnetic.

Waves satisfying the latter condition together with $\mathbf{E} \perp \mathbf{M}$ are called *pure* waves. The density of electromagnetic energy, $\frac{1}{2}KE^2 + M^2$, now becomes

$$u = KE^2 = M^2 . \quad . \quad . \quad . \quad . \quad (8)$$

In order to obtain the velocity of propagation v, eliminate from (5) either $\mathbf{M}$ or $\mathbf{E}$. Thus, remembering that $\mathbf{n}^2 = 1$,

$$\frac{v^2K}{c^2}\mathbf{E} = \mathrm{V}\mathbf{n}\mathrm{V}\mathbf{En} = \mathbf{E} - (\mathbf{En})\mathbf{n},$$

and since $\mathbf{En} = 0$,

$$v = \frac{c}{\sqrt{K}}, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

the well-known result.

Finally, the flux of energy can be written, again by the second of equations (5),

$$\mathbf{F} = \frac{c^2}{v}\mathrm{V}\mathbf{E}\mathrm{V}\mathbf{nE} = \frac{c^2E^2}{v}\mathbf{n},$$

i.e. by (8) and (9),

$$\mathbf{F} = u \,.\, v\mathbf{n} = u\mathbf{v}, \quad . \quad . \quad . \quad . \quad (10)$$

where $\mathbf{v}$ is the velocity of propagation in magnitude and direction.

This simple formula can be read: the electromagnetic

energy is carried forward with the vector velocity **v**, as if it were a fluid of density u.

4. Reflection and Refraction at the Boundary of Isotropic Media; E in Plane of Incidence.

Since any vibrations of the electric vector **E** (and of its magnetic companion) can always be split into two rectilinear vibrations in two mutually perpendicular directions, it is possible, and convenient, to treat separately the two cases of monochromatic plane waves of *rectilinearly polarized* incident light: 1st, **E** parallel, and 2nd, **E** perpendicular to the plane of incidence. As concerns **M** we know already that it is entirely determined by **E** and by the direction of the ray, its intensity being given by $M^2 = KE^2$, and its direction by the circumstance that, in each of the adjacent media, **E**, **M**, **ray** is a right-handed orthogonal system of vectors.

Let us begin with the first case, **E** in *plane of incidence*. Let the interface of the two isotropic media, whose permittivities (corresponding to the given frequency) are K and K', be a plane. Take this as the y, z plane, and let the normal to the interface drawn towards the first medium (K) be the axis of positive x. Let the plane waves arrive from the first medium (K) towards the second (K'). It will be convenient to take the z-axis along the intersection of one of the wave planes with the boundary. Then x, y will be the plane of incidence. Denoting the angle of incidence by α, the angle of reflection by α_1, and the angle of refraction by β (Fig. 1), use the abbreviations

$$\begin{aligned} a &= -\cos\alpha, & a_1 &= \cos\alpha_1, & a' &= -\cos\beta \\ b &= -\sin\alpha, & b_1 &= -\sin\alpha_1, & b' &= -\sin\beta. \end{aligned}$$

Then the distances measured along the incident, the reflected, and the refracted rays will be

$$ax + by, \qquad a_1x + b_1y, \qquad a'x + b'y,$$

respectively. If, therefore, v and v' be the velocities of propagation in the first and the second medium, *i.e.* by (9),

$$= \frac{c}{\sqrt{K}} \qquad v' = \frac{c}{\sqrt{K'}}$$

then **E**, **M** will be simple periodic functions of the arguments $ax + by - vt$ for the incident, $a_1x + b_1y - vt$ for the reflected, and $a'x + b'y - v't$ for the refracted wave. Thus, g and g' being constant factors, the vectors **E** and **M** will be proportional to the exponential functions

$$\left.\begin{array}{lr} \text{exp. } ig(ax + by - vt) & \text{in the incident ray} \\ \text{exp. } ig(a_1x + b_1y - vt) & \text{,, reflected ray} \\ \text{exp. } ig'(a'x + b'y - v't) & \text{,, refracted ray} \end{array}\right\} \quad . \qquad (11)$$

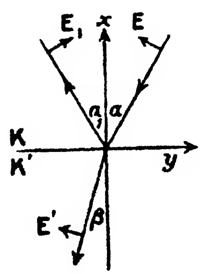

FIG. 1.

The electric force being in the plane of incidence (plane of Fig. 1), the magnetic force will be parallel to the axis of z. Let its intensity, taken positive along the positive z-axis, be denoted by M for the incident, by M_1 for the reflected, and by M' for the refracted wave. Let E, E_1, E' be the corresponding symbols for the electric forces, taken positive in the direction of the arrows. Then, by the second of equations (5),

$$E=\frac{v}{c}M,\quad E_1=\frac{v}{c}M_1,\quad E'=\frac{v'}{c}M' \quad . \quad . \quad (12)$$

The differential equations (1) being now all satisfied, for each wave separately, it remains only to take account of the *boundary conditions*. These are, in virtue of the equations (1) themselves, as is well known:—

1st, the continuity of the whole magnetic force **M**,*

2nd, the continuity of the tangential component of the electric force **E**, and

3rd, the continuity of the normal component of the dielectric displacement **D** = K**E**.

Thus we have, for $x=0$,

$$M+M_1=M', \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$
$$E\cos\alpha - E_1\cos\alpha_1 = E'\cos\beta \quad . \quad . \quad . \quad (b)$$
$$K(E\sin\alpha + E_1\sin\alpha_1) = K'E'\sin\beta \quad . \quad . \quad (c)$$

Since these conditions are to be satisfied for every y and for all times t, we have in the first place, by (11), prior to any considerations concerning the amplitudes,

$$gb = gb_1 = g'b',$$
$$gv = g'v',$$

or, remembering the meaning of b, etc.,

$$\left.\begin{aligned} \alpha_1 &= \alpha, \\ \sin\alpha : \sin\beta &= v : v' = n \end{aligned}\right\} \quad . \quad . \quad . \quad (13)$$

i.e. the two fundamental laws of geometrical optics: *equality of angles and identity of planes of incidence and of reflection*, and *Snellius' law of refraction*. At the same time we have, by (9), for the ratio n, independent of α, *i.e.* for what is called the *refractive index* of the second medium relatively to the first,

* Since $\mu=\mu'=1$.

$$n = \sqrt{\frac{K'}{K}} \quad . \quad . \quad . \quad . \quad (14)$$

In virtue of (12), the third boundary condition (c) now becomes identical with (a). Thus but two conditions are left which, by (12), can be written entirely in terms of the electric forces,

$$E + E_1 = \frac{v}{v'} E' \quad . \quad . \quad . \quad . \quad (a)$$

$$E - E_1 = \frac{\cos\beta}{\cos\alpha} E' \quad . \quad . \quad . \quad (b)$$

Eliminate E'. Then

$$\frac{E_1}{E}\left(\frac{\sin\beta}{\sin\alpha} + \frac{\cos\alpha}{\cos\beta}\right) = \frac{\cos\alpha}{\cos\beta} - \frac{\sin\beta}{\sin\alpha}.$$

Add (a) to (b) and use the above refraction law, (13). Then the result will be

$$\frac{E'}{E}\left(\frac{\sin\alpha}{\sin\beta} + \frac{\cos\beta}{\cos\alpha}\right) = 2.$$

After an easy trigonometrical transformation we have

$$\left.\begin{aligned} \frac{E_1}{E} &= \frac{\tan(\alpha - \beta)}{\tan(\alpha + \beta)} \\ \frac{E'}{E} &= \frac{2\cos\alpha\sin\beta}{\sin(\alpha+\beta)\,.\,\cos(\alpha-\beta)} \end{aligned}\right\} \quad . \quad . \quad (15)$$

These are the famous *formulæ of Fresnel* for incident light whose "*plane of polarization*" *is normal to the plane of incidence*.

According to Fresnel the vibrations of the elastic æther were *perpendicular to the plane of polarization*. In the present case, therefore, the Fresnelian vibrations would be parallel to the plane of incidence. On the other hand,

according to Neumann the æther vibrated *in the plane* of polarization. We thus see that

Fresnel's light-vector * corresponds to **E** (or **D**),
Neumann's light-vector corresponds to **M**.

The distinction between **E** and **D** remains immaterial until we come to treat anisotropic media.

The electromagnetic theory of light has thus united in itself, as far at least as reflection and refraction are concerned, both theories, Fresnel's and Neumann's, and the famous quarrel about the direction of æthereal vibrations with respect to the plane of polarization has, in its original sense, become an idle controversy. The problem has now acquired a different meaning, *viz.*: is the action of light to be ascribed to the electrical or to the (inseparable) magnetic oscillations? The experiments of O. Wiener † on stationary light waves are supposed to speak in favour of the *electric* vector, as far at least as the action upon photographic plates or the excitation of fluorescence are concerned.

The ratio of E_1 to E has, by (15), a positive value. Since, however, the positive senses of E, E_1 have been taken along the arrows of Fig. 1, the positive sign of the ratio signifies that the tangential component of the electric force undergoes a change of phase by 180° on being reflected. The phase of the normal component is not changed. Also the refracted ray proceeds without change of phase.

The *intensity* of light being measured by the time-average of u or of $KE^2 = M^2$, and the oscillations being periodic, the intensities I, I_1, I′ will be given by the squared amplitudes of E, E_1, E' multiplied by K, K_1, K' respectively, *i.e.*:

$$\frac{\mathrm{I}_1}{\mathrm{I}} = \left(\frac{E_1}{E}\right)^2, \quad \frac{\mathrm{I}'}{\mathrm{I}} = n^2\left(\frac{E'}{E}\right)^2 \qquad . \qquad . \qquad (16)$$

In the case of *normal* incidence, *i.e.* for $\alpha = \beta = 0$, we have

* That is, the vibrating or periodically variable directed magnitude.
† Started in 1890. See *Ann. der Physik*, vol. xl., p. 203.

from (15), or simpler, returning to the original equations (*a*), (*b*) which now become

$$E + E_1 = nE'; \quad E - E_1 = E',$$

$$\frac{E_1}{E} = \frac{n-1}{n+1}; \quad \frac{E'}{E} = \frac{2}{1+n} \quad . \quad . \quad . \quad (15^a)$$

Notice that in this case $E_1 + E' = E$, an obvious property.

Returning to the first of the general formulæ (15) we see that for

$$\alpha + \beta = \pi/2$$

$E_1 = 0$, *i.e.* light polarized normally to the plane of incidence is not reflected at all. A glance upon Fig. 1 suffices to see that this happens when the reflected ray would be perpendicular to the refracted one. This is *Brewster's law.* The corresponding angle of incidence α_0, called *the angle of polarization*, is determined by

$$n = \frac{\sin \alpha_0}{\sin \beta_0} = \frac{\sin \alpha_0}{\sin\left(\frac{\pi}{2} - \alpha_0\right)}.$$

Thus

$$\tan \alpha_0 = n \quad . \quad . \quad . \quad . \quad (17)$$

It can be easily verified that the light incident at the angle of polarization penetrates *entirely* into the second medium, *i.e.* that $I' = I$. In fact, for $\alpha = \alpha_0$, the second of (15) becomes

$$\frac{E'}{E} = \frac{2\cos^2 \alpha_0}{\cos(2\alpha_0 - \frac{1}{2}\pi)} = \cot \alpha_0 = \frac{1}{n},$$

and therefore, by (16),

$$I' : I = n^2\left(\frac{E'}{E}\right)^2 = 1,$$

in agreement with the principle of conservation of energy.

It may be well to mention that Fresnel's formula $E_1 : E = \tan(\alpha - \beta) : \tan(\alpha + \beta)$, showing in general a very good agreement with observation, deviates somewhat from reality in the neighbourhood of the angle of polarization. It was found that these slight deviations are influenced by external circumstances * which produce at the reflecting face a thin optically heterogeneous sheet. In fact, it has been possible to account for these deviations theoretically by assuming such a transitional layer.

6. Reflection and Refraction; E ⊥ Plane of Incidence. Note on the Transition Layer.

In this case the electric vector is parallel to the boundary of the two media, *i.e.* to the axis of z, while the magnetic vector is contained in the plane of incidence. In the habitual terminology, the incident light is polarized in the plane of polarization.

Proceeding as before, *i.e.* writing down the boundary conditions which now require the continuity of the whole electric force and of the whole magnetic force, we get again $\alpha_1 = \alpha$, $\sin\alpha : \sin\beta = n$, and for the ratios of the electric amplitudes,

$$\left.\begin{aligned} \frac{E_1}{E} &= -\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)} \\ \frac{E'}{E} &= \frac{2\cos\alpha\sin\beta}{\sin(\alpha+\beta)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \qquad (18)$$

These formulæ are again identical with those known as Fresnel's formulæ for incident light *polarized in the plane of incidence*. The electric force again corresponds to Fresnel's light-vector.

* Thus, for instance, according to Drude, the deviations from Fresnel's formula for a freshly broken face of a crystal of rock-salt were very small, but, the reflecting face being exposed to the action of air, the deviations began to increase rapidly.

For normal incidence formulæ (18) become, in appearance only, indeterminate. Returning to the original form of the boundary conditions, the reader will obtain for $\alpha = 0$, without trouble,

$$\frac{E_1}{E} = -\frac{n-1}{n+1}, \quad \frac{E'}{E} = \frac{2}{n+1} \quad . \quad . \quad (18^{\circ})$$

From the first of (18) we see that E_1/E never vanishes, *i.e.* that light polarized in the plane of incidence is not extinguished at any angle of reflection. In the preceding section we saw that light polarized perpendicularly to the plane of incidence is not reflected at all when α becomes equal to the polarization angle $\alpha_0 = \text{arc tan } n$. Thus, common or natural light (which can be considered as consisting of both the above kinds), when reflected under the angle α_0, would be polarized in the plane of incidence, *i.e.* so that only the electric oscillations, normal to the plane of incidence, would remain. According to the above Fresnelian formulæ this polarization would be complete, while experiments give a slight residue of electric vibrations contained in the plane of incidence. These small deviations of theory from observation can, however, as was already remarked, be accounted for by assuming a heterogeneous layer of transition, *i.e.* having a permittivity κ which in the narrow limits

$$x = -\epsilon \text{ to } x = \epsilon$$

depends upon x, and assumes for $x \leqq -\epsilon$ and $x \geqq \epsilon$ the constant values K' and K respectively.

Drude * has shown that an approximate treatment, in which the integral values

* *Cf.* Drude's *Lehrbuch der Optik*, second edition, Leipzig, 1906, pp. 272-80, or English translation by Mann and Millikan, 1913, Longmans, Green & Co.

$$\int_{-\epsilon}^{\epsilon} \kappa dx, \quad \int_{-\epsilon}^{\epsilon} \frac{dx}{\kappa}$$

are the only relevant ones, is practically sufficient. If the ray, incident under the angle a_0 = arc tan n, is rectilinearly polarized in a plane oblique to the plane of incidence, say under the azimuth ζ of 45°, then the reflected light, instead of being polarized rectilinearly contains also traces of *elliptic* * polarization, in agreement with observation. The ellipse resulting from Drude's theoretical investigation is very long, the ratio ρ of the minor to the major axis being (for $\zeta = 45°$)

$$\rho = \frac{\pi}{\lambda} \frac{\sqrt{K + K'}}{K - K'} \int_{-\epsilon}^{\epsilon} \frac{(\kappa - K)(\kappa - K')}{\kappa} dx, \quad . \quad (19)$$

where λ is the wave-length (in vacuo). The absolute value of the integral does not exceed that which would correspond to $\kappa = \sqrt{KK'}$ = const. Thus we have from (19), for the upper limit of the *thickness* $l = 2\epsilon$ of the transition layer,

$$\frac{l}{\lambda} = \frac{|\rho|}{\pi\sqrt{1 + n^2}} \cdot \frac{n + 1}{n - 1} \quad . \quad . \quad . \quad (20)$$

Now, the observed value of ρ for heavy flint glass in contact with air (n = 1·75 for sodium light) is

$$\rho = 0{\cdot}03,$$

this being the highest ratio of axes yet observed. In this case formula (20) would give l/λ = 0·0174, *i.e.* l = 1·023 . 10^{-5} mlm. In other cases we would have obtained thicknesses even a hundred times smaller. In fact, for other kinds of glass, of smaller refractive index, values of ρ have been found scarcely exceeding 0·007, and Lord Rayleigh's value for water, whose

* The ellipticity being due to the difference of phase of the component oscillations.

surface has been carefully cleaned, was as small as 0·00035. The corresponding thickness of the layer of transition would be nearly as small as 10^{-7} mlm., that is, of the order of molecular dimensions.

6. Total Reflection.

Let the "first" medium, from which the light arrives, be optically denser than the "second," *i.e.* in the above symbols,

$$\frac{K}{K'} > 1; \quad n < 1.$$

Then, for $a = \omega = \text{arc} \sin n$,

$$\sin \beta = 1, \quad \beta, = \frac{\pi}{2},$$

and for $a > \omega$. $\qquad \sin \beta > 1.$

That is, β will cease to be a real angle. Thus for $a > \omega$ we shall have the well-known phenomenon of *total reflection.* Theoretically there will exist, for $a > \omega$, a refracted wave, whose amplitude, however, in penetrating the second, thinner medium, will die away exponentially, the more rapidly the greater the difference $a - \omega$. In fact, returning to (11), and remembering that $a' = -\cos \beta$, $b' = -\sin \beta$, notice that the forces in the refracted ray are proportional to

$$e^{-ig'(x \cos \beta + y \sin \beta + v't)},$$

where $\qquad \cos \beta = \pm \sqrt{1 - \frac{\sin^2 a}{n^2}}$

is imaginary (since $\sin a > n$). Write, therefore,

$$\cos \beta = \pm i\gamma,$$

where $\gamma = \sqrt{\frac{\sin^2 \alpha}{n^2} - 1}$ is real. Thus the part of the above exponential depending on x will become $e^{\pm \gamma g' x}$. The minus sign is, by physical reasons, to be rejected, since it would give an indefinitely increasing amplitude of waves penetrating into the thinner medium (x negative). What remains, therefore, is

$$e^{\gamma g' x} = e^{-\gamma g' x'},$$

where $x' = -x$. If T be the period of oscillations and λ' the wave-length in the thinner medium, $g'v' = 2\pi/\mathrm{T}$, $g' = 2\pi/\lambda'$, so that the above damping factor becomes

$$e^{-2\pi\gamma x'/\lambda'}.$$

Thus, if $\alpha - \omega$ is sensibly positive, the amplitude of the refracted vibrations will become *evanescent in a depth of a few wave-lengths.*

Notice that besides this factor we have only

$$e^{-ig'(y \sin \beta + v't)},$$

so that these rapidly damped oscillations are propagated in the second medium along the interface, *viz.* in the direction of the negative y-axis.

Let us now see what the properties of the *reflected* waves are, after the "limiting angle"

$$\alpha = \omega = \text{arc} \sin n$$

has been exceeded.

Take, for example, the case in which the incident light is rectilinearly polarized under the angle $\zeta = 45°$ relatively to the plane of incidence. Let P be the component of $\mathbf{E}$ in the plane of incidence, and Z its component along the z-axis. Let P_1, Z_1 have similar meanings for the reflected wave. Thus P_1 and Z_1 will be what has been denoted by E_1 in (15)

and in (18) respectively, and since (for $\zeta = 45°$) P = Z, we shall have

$$\frac{P_1}{Z_1} = -\frac{\tan(\alpha-\beta)}{\tan(\alpha+\beta)}\cdot\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = -\frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}.$$

Since $\sin\beta > 1$, this ratio will have a complex value, say $\rho e^{i\Delta}$ where ρ and Δ are real. The former will be the ratio of the absolute values $|P_1|$, $|Z_1|$ or the ratio of the amplitudes, and the latter the phase difference of the component contained in the plane of incidence and the component normal to this plane. Developing trigonometrically the last expression, remembering that $\sin\beta = \frac{1}{n}\sin\alpha$, and that

$$\cos\beta = i\gamma, \quad \gamma = \frac{1}{n}\sqrt{\sin^2\alpha - n^2}, \quad . \quad . \quad (21)$$

the reader will find

$$\frac{P_1}{Z_1} = \rho e^{i\Delta} = \frac{(\sin^2\alpha - i\,.\,n\gamma\cos\alpha)^2}{\sin^4\alpha + n^2\gamma^2\cos^2\alpha},$$

whence

$$\rho = 1, \quad i.e.\ |P_1| = |Z_1|, \quad . \quad . \quad . \quad (22)$$

and using (21),

$$\tan\frac{\Delta}{2} = \frac{\cos\alpha\sqrt{\sin^2\alpha - n^2}}{\sin^2\alpha} \quad . \quad . \quad (23)$$

Thus the amplitudes of the reflected components P_1, Z_1 will be equal to one another,* while their phases will differ

* The reader will easily show that these reflected amplitudes are equal to the incident ones, *i.e.* (whether ζ is 45° or not)

$$\text{ampl. } P_1 = \text{ampl. } P, \qquad \text{ampl. } Z_1 = \text{ampl. } Z.$$

This result could be expected, without returning to (15) and (18). In

from one another, the more so the more the limit angle ω is exceeded. The reflected light, for any $a > \omega$, will be *elliptically* polarized. This is a general property of total reflection which in its essence has already been mastered by Fresnel. If $\zeta \neq 45°$, then, instead of $\rho = 1$,

$$\rho \equiv \frac{|P_1|}{|Z_1|} = \frac{P}{Z},$$

while the phase difference continues to obey the formula (23).

The azimuth 45° is of particular interest because it gives for $\Delta = 90°$ a *circularly* polarized reflected ray.

In order to obtain $\Delta = 45°$ we have to make, by (23),

$$\frac{\cos a}{\sin^2 a}\sqrt{\sin^2 a - n^2} = \tan 22{\cdot}5°,$$

that is, for glass of refractive index $\frac{1}{n} = 1{\cdot}51$, say, in contact with air, either $a = 48° \, 37'$ or $a = 54° \, 37'$. By *two* reflections under any of these angles * the phase difference $\Delta = \frac{\pi}{2}$, *i.e.* circularly polarized light is obtained. This is the principle of the universally known glass rhomboëder of Fresnel ($a \doteqdot 54°$). Notice that by a single reflection the phase difference of $\frac{\pi}{2}$ could not be obtained since, for $\frac{1}{n} = 1{\cdot}51$, the *maximum* of Δ amounts, by (23), only to 45° 36′.

fact, it is enough to remember that in the thinner medium ($x < 0$) the electromagnetic energy flows only *along* the interface and not normally to it.

* Both are greater than the limit angle ω, as is necessary for total reflection; in fact, $\omega = \text{arc}\sin\frac{1}{1{\cdot}51} = 41° \, 26'$.

7. Optics of Crystalline Media: General Formulæ and Theorems.

The propagation of plane waves of monochromatic light in a crystalline medium obeys the formulæ (5) deduced in Section 3,

$$\frac{v}{c}\mathbf{D} = \mathrm{V}\mathbf{M}\mathbf{n}\,; \quad \frac{v}{c}\mathbf{M} = \mathrm{V}\mathbf{n}\mathbf{E} \quad . \quad . \quad . \quad (5)$$

Here $\mathbf{n}$ is the wave-normal, a unit vector, and v the velocity of propagation of the waves. The dielectric displacement

$$\mathbf{D} = K\mathbf{E}$$

is a linear vector function of the electric force $\mathbf{E}$.

The whole optics of a homogeneous crystal is contained in the two simple equations (5), the properties of the crystal being defined by the vector operator K.

As was already mentioned, formulæ (5) contain the fundamental relations

$$\mathbf{M}\mathbf{n} = 0, \quad \mathbf{D}\mathbf{n} = 0,$$

i.e. the magnetic force and the dielectric displacement lie in the wave-plane, or are purely *transversal*. Notice that what corresponds to Fresnel's light-vector is the dielectric displacement $\mathbf{D}$, and not the electric force $\mathbf{E}$.

The latter is in general not contained in the wave-plane, *i.e.* $\mathbf{E}\mathbf{n} \neq 0$.

Formulæ (5) give also the fundamental relations

$$\mathbf{M}\mathbf{D} = 0, \quad \mathbf{M}\mathbf{E} = 0.$$

Thus the magnetic force is always normal to both the electric force and the displacement.

The *ray*, defined by the direction of the energy flux

$$\mathbf{F} = c\mathrm{V}\mathbf{E}\mathbf{M},$$

is always normal to **E**, **M**, but in general *oblique* relatively to the wave-plane.

Multiplying the first of (5) by **E**, and remembering that **EVMn** = **MVnE**, we have

$$\mathbf{ED} = M^2 \quad . \quad . \quad . \quad . \quad (24)$$

of which equation (7) is but a special case. Thus the luminous energy, in a crystal as well as in an isotropic medium, consists in equal parts of electric and of magnetic energy.

In order to obtain an equation for the velocity of propagation v as dependent upon the direction of the wave-normal **n**, eliminate from (5) the magnetic force. Then, remembering that $\mathrm{V}\mathbf{A}\mathrm{V}\mathbf{BC} = \mathbf{B}(\mathbf{CA}) - \mathbf{C}(\mathbf{AB})$, and that $\mathbf{n}^2 = 1$,

$$\frac{v^2}{c^2}\mathbf{D} = \mathrm{V}\mathbf{n}\mathrm{V}\mathbf{En} = \mathbf{E} - (\mathbf{En})\mathbf{n},$$

and since $\mathbf{D} = K\mathbf{E}$,

$$\left[1 - \frac{v^2}{c^2}K\right]\mathbf{E} = (\mathbf{En})\mathbf{n},$$

where the bracketed expression, as well as K itself, is a linear vector operator. The last equation will conveniently be written

$$\mathbf{D} = K\mathbf{E} = (\mathbf{En})\left[\frac{K}{1 - \frac{v^2}{c^2}K}\right]\mathbf{n}, \quad . \quad . \quad (25)$$

where the bracketed expression is again a linear vector operator, having the same principal axes as K and the corresponding principal values

$$\frac{K_1}{1 - \frac{v^2}{c^2}K_1}, \quad \frac{K_2}{1 - \frac{v^2}{c^2}K_2}, \quad \frac{K_3}{1 - \frac{v^2}{c^2}K_3},$$

K_1, K_2, K_3 being the principal values of K.

Now, $\mathbf{Dn} = 0$. Thus multiplying (25) scalarly by $\mathbf{n}$ and remembering that the scalar product $(\mathbf{En})$ is in general different from zero, we have, for the velocity of propagation v, the equation

$$\mathbf{n}\left[\frac{1}{v^2 - c^2/K}\right]\mathbf{n} = 0, \qquad (26)$$

which is but the vectorial form of the famous *Fresnelian equation.* In fact, denote by n_1, n_2, n_3 the components of the wave-normal $\mathbf{n}$, or its direction cosines, with respect to the *principal electrical axes* of the crystal, and use the abbreviations

$$c^2/K_1 = v_1^2, \quad c^2/K_2 = v_2^2, \quad c^2/K_3 = v_3^2.$$

Then the Cartesian expansion of (26) will be

$$\frac{n_1^2}{v^2 - v_1^2} + \frac{n_2^2}{v^2 - v_2^2} + \frac{n_3^2}{v^2 - v_3^2} = 0 \qquad (26a)$$

which is Fresnel's equation.

For any given direction of the normal $\mathbf{n}$ we have, from (26) or (26a), in general *two different* absolute values of v, say v' and v''. Let the vectors $\mathbf{E}$, $\mathbf{D}$ corresponding to v' and to v'' be $\mathbf{E}'$, $\mathbf{D}'$ and $\mathbf{E}''$, $\mathbf{D}''$, respectively. Then, by (25),

$$\mathbf{D}' = (\mathbf{E}'\mathbf{n})\left[\frac{c^2}{c^2/K - v'^2}\right]\mathbf{n},$$

$$\mathbf{D}'' = (\mathbf{E}''\mathbf{n})\left[\frac{c^2}{c^2/K - v''^2}\right]\mathbf{n}.$$

If, therefore, the waves are to be propagated with the same velocity (either v' or v''), *i.e.* if they are not to split into two trains of waves, then the vector $\mathbf{D}$ *must have one of the two directions*

$$\mathbf{s}' = \left[\frac{1}{v'^2 - c^2/K}\right]\mathbf{n} \text{ or } \mathbf{s}'' = \left[\frac{1}{v''^2 - c^2/K}\right]\mathbf{n} \qquad (27)$$

where the $\pm$ signs, being, of course, irrelevant, have been omitted. These two privileged directions $\mathbf{s}'$ and $\mathbf{s}''$ of light oscillations, belonging to a given normal $\mathbf{n}$, are *perpendicular* to one another.

In fact, denoting the vector operator in (26) by ω, we can write Frésnel's equation

$$\mathbf{n}\omega\mathbf{n} = 0, \quad . \quad . \quad . \quad . \quad (26b)$$

and, instead of (27),

$$\mathbf{s}' = \omega'\mathbf{n}, \quad \mathbf{s}'' = \omega''\mathbf{n},$$

and since ω, and therefore ω', ω'', are *self-conjugate* operators,* and have the same principal axes, we have

$$\mathbf{s}'\mathbf{s}'' = \mathbf{s}' . \omega''\mathbf{n} = \mathbf{n} . \omega''\mathbf{s}' = \mathbf{n} . \omega''\omega'\mathbf{n}.$$

But

$$\frac{1}{\omega'} - \frac{1}{\omega''} = v'^2 - v''^2, \ i.e.\ \omega''\omega' = \omega'\omega'' = \frac{\omega'' - \omega'}{v'^2 - v''^2},$$

and, therefore,

$$\mathbf{s}'\mathbf{s}'' = \frac{\mathbf{n}\omega''\mathbf{n} - \mathbf{n}\omega'\mathbf{n}}{v'^2 - v''^2}.$$

By (26b) both terms in the numerator vanish.

Thus, if only $v' \neq v''$,

$$\mathbf{s}'\mathbf{s}'' = 0, \quad . \quad . \quad . \quad (28)$$

that is, $\mathbf{s}' \perp \mathbf{s}''$. Q.E.D.

The case $v' = v''$, which occurs only for certain directions of the normal $\mathbf{n}$, will be considered a little later on.

If the vector $\mathbf{D}$ has neither of these two privileged direc-

* For so is also the permittivity-operator K.

tions, then the waves are *split* into two separate trains. In this case, **D** being purely transversal, it can always be split into two components, D' along $\mathbf{s}'$ and D'' along $\mathbf{s}''$. One train of waves will carry away with the velocity v' the component D' of the dielectric displacement, and another, with velocity v'', the component D''. It is this which constitutes what is called *double refraction* in crystals.

The usual form of the equation for the direction cosines of Fresnel's light-vector can easily be obtained from (27). In fact, writing summarily $\mathbf{s}$ for $\mathbf{s}'$ or $\mathbf{s}''$ and denoting by s_1, s_2, s_3 the components of $\mathbf{s}$ along the principal electrical axes of the crystal, we have

$$\frac{n_1}{s_1} = v^2 - c^2/K_1 = v^2 - v_1^2, \quad \frac{n_2}{s_2} = v^2 - v_2^2, \quad \frac{n_3}{s_3} = v^2 - v_3^2,$$

whence the required relation

$$\frac{n_1}{s_1}(v_2^2 - v_3^2) + \frac{n_2}{s_2}(v_3^2 - v_1^2) + \frac{n_3}{s_3}(v_1^2 - v_2^2) = 0; \qquad (29)$$

s_1, etc., are proportional to the direction cosines of the vector **D** which corresponds to Fresnel's light-vector.

In order to determine the direction of the *ray*, *i.e.* that of the energy flux $\mathbf{F} = c\mathrm{V}\mathbf{EM}$, return once more to the equations (5). The second of these gives

$$\mathbf{F} = \frac{c^2}{v}\mathrm{V}\mathbf{E}\mathrm{V}\mathbf{nE} = \frac{c^2}{v}[E^2\mathbf{n} - (\mathbf{En})\mathbf{E}];$$

now, by (26), writing again $\omega = [v^2 - c^2/K]^{-1}$,

$$\mathbf{E} = -c^2(\mathbf{En})\frac{\omega}{K}\mathbf{n}; \qquad (30)$$

thus,

$$\mathbf{F} = \frac{c^4}{v}(\mathbf{En})^2\left[c^2\left(\frac{\omega}{K}\mathbf{n}\right)^2_\bullet \cdot \mathbf{n} + \frac{\omega}{K}\mathbf{n}\right].$$

If, therefore, **p** is a unit vector along the luminous *ray* and σ a scalar to be determined presently,

$$\frac{1}{\sigma}\mathbf{p} = \left(\frac{\omega}{K}\mathbf{n}\right)^2\mathbf{n} + \frac{1}{c^2}\frac{\omega}{K}\mathbf{n}.$$

In order to obtain the value of σ, square the last equation, remembering that $\mathbf{p}^2 = 1$, $\mathbf{n}^2 = 1$, and that, by (30),

$$\mathbf{n}\frac{\omega}{K}\mathbf{n} = -\frac{1}{c^2}. \qquad (31)$$

Then the result will be

$$\frac{1}{\sigma^2} = \left(\frac{\omega}{K}\mathbf{n}\right)^4 - \frac{1}{c^4}\left(\frac{\omega}{K}\mathbf{n}\right)^2.$$

Ultimately, therefore, the ray will be given by

$$\mathbf{p} = \frac{c^2W^2\mathbf{n} + \mathbf{W}}{W\sqrt{c^4W^2 - 1}} \qquad (32)$$

where

$$\mathbf{W} = \frac{\omega}{K}\mathbf{n} = \left[\frac{1}{v^2K - c^2}\right]\mathbf{n}.$$

According as **D** vibrates along $\mathbf{s}' = \omega'\mathbf{n}$ or along $\mathbf{s}'' = \omega''\mathbf{n}$, we have to substitute in (32) $v = v'$ or $v = v''$, and therefore $\mathbf{W} = \mathbf{W}'$ or $\mathbf{W}''$. In general, therefore, for any wave-normal **n** we shall have *two different rays*, $\mathbf{p}'$ and $\mathbf{p}''$.

Remembering that $\mathbf{pn} = \cos(\mathbf{p}, \mathbf{n})$ and that, by (31), $\mathbf{Wn} = -\frac{1}{c^2}$, we have, for the angle contained between the ray and the wave-normal,

$$\cos(\mathbf{p}, \mathbf{n}) = \sqrt{1 - \frac{1}{c^4W^2}} \qquad (33)$$

Thus the angle $(\mathbf{p}, \mathbf{n})$, *i.e.* each of the angles $(\mathbf{p}', \mathbf{n})$, $(\mathbf{p}'', \mathbf{n})$, is in general different from zero, or the two rays $\mathbf{p}'$, $\mathbf{p}''$ generally deviate from the wave-normal.

8. The Properties of the Electrical Axes of a Crystal.

The principal electrical axes of the crystal, *i.e.* the principal axes of the operator K, being mutually perpendicular, let us take along them the three normal unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. Let, for instance, $\mathbf{i}$ coincide with that axis to which corresponds the smallest, and $\mathbf{k}$ with that to which corresponds the greatest principal value of K. In short, let

$$K_3 > K_2 > K_1,$$

or, writing again $c^2/K_i = v_i^2$,

$$v_1 > v_2 > v_3 \qquad . \quad . \quad . \quad . \quad (34)$$

This order of arrangement which, for the moment, is irrelevant will be referred to in the next section.

If the wave-normal coincides with one of the principal electrical axes, *i.e.* if $\mathbf{n} = \mathbf{i}$ or $\mathbf{j}$ or $\mathbf{k}$, we have from Fresnel's equation (26*a*), remembering that for $\mathbf{n} = \mathbf{i}$, $n_2 = n_3 = 0$, $n_1 = 1$, etc., the following velocities of propagation

$$\left.\begin{array}{llll} \text{for } \mathbf{n} = & \mathbf{i}, & \mathbf{j}, & \mathbf{k} \\ v' = & v_2, & v_3, & v_1 \\ v'' = & v_3, & v_1, & v_2 \end{array}\right\} \qquad . \quad . \quad . \quad (35)$$

Thus, $v_1 = c/\sqrt{K_1}$, $v_2 = c/\sqrt{K_2}$, $v_3 = c/\sqrt{K_3}$ are the so-called *principal velocities* of propagation. If the normal is along the *first* electrical axis, the waves are propagated with the *second* or the *third* principal velocity (according to the direction of $\mathbf{D}$), and so on, by cyclic permutation.

In order to determine the directions $\mathbf{s}'$, $\mathbf{s}''$ in which $\mathbf{D}$ must

vibrate in each of the above three cases if the wave is not to split, return * to the equation

$$\frac{v^2}{c^2}\mathbf{D} = \mathbf{E} - (\mathbf{En})\mathbf{n},$$

already obtained from (5), p. 36. This gives, for $\mathbf{n} = \mathbf{i}$,

$$\frac{v^2}{c^2}\mathbf{D} = E_2\mathbf{j} + E_3\mathbf{k},$$

and, therefore, by (35),

$$\mathbf{D}' = K_2(E_2'\mathbf{j} + E_3'\mathbf{k}).$$

But $\mathbf{D}' = K\mathbf{E}'' = K_1E_1'\mathbf{i} + K_2E_2'\mathbf{j} + K_3E_3'\mathbf{k}$. Thus $E_1' = 0$, and since $K_2 \neq K_3$,

$$\mathbf{D}' = K_2E_2'\mathbf{j}.$$

Similarly we shall find $\mathbf{D}'' = K_3E_3''\mathbf{k}$. Analogous relations will take place for $\mathbf{n} = \mathbf{j}, \mathbf{k}$. Thus, denoting by $\|$ the parallelism of vectors, we have

$$\left.\begin{array}{rlll} \text{for } \mathbf{n} = & \mathbf{i}, & \mathbf{j}, & \mathbf{k} \\ \mathbf{D}' \,\| & \mathbf{j}, & \mathbf{k}, & \mathbf{i} \\ \mathbf{D}'' \,\| & \mathbf{k}, & \mathbf{i}, & \mathbf{j} \end{array}\right\} \quad . \quad . \quad . \quad (35a)$$

These relations, together with (35), can easily be expressed in words. In passing we have also seen that the electric force $\mathbf{E}$, being in general oblique, becomes for $\mathbf{n} = \mathbf{i}, \mathbf{j}, \mathbf{k}$, purely transversal, *i.e.* falls into the wave-plane. And since $\mathbf{M}$ is always in the wave-plane, the ray $\mathbf{p}$ coincides with the normal $\mathbf{n}$ when this coincides with any of the principal electrical axes of the crystal.

* Since (27) or (25) becomes in the present case indeterminate owing to the fact that $\mathbf{En}$ will now vanish.

9. Optical Axes.

Returning to Fresnel's equation (26*a*) for the velocity of propagation, use the abbreviations

$$\left.\begin{aligned}\alpha &= n_1^2(v_2^2 - v_3^2)\\ \beta &= n_2^2(v_3^2 - v_1^2)\\ \gamma &= n_3^2(v_1^2 - v_2^2)\end{aligned}\right\} \quad . \quad . \quad . \quad (36)$$

Then

$$2v^2 = n_1^2(v_2^2 + v_3^2) + n_2^2(v_3^2 + v_1^2) + n_3^2(v_1^2 + v_2^2) \pm \sqrt{(\alpha + \beta - \gamma)^2 - 4\alpha\beta} \quad . \quad . \quad (37)$$

Let our previous v' correspond to the upper, and v'' to the lower sign of the square root.

Let us find those particular directions of the wave-normal **n** for which the two velocities of propagation become equal,

$$v' = v''.$$

The necessary and sufficient condition for this equality is

$$(\alpha + \beta - \gamma)^2 - 4\alpha\beta = 0.$$

But, by (34), α is positive, β negative, and γ positive. Thus, we must have, separately,

$$\alpha + \beta - \gamma = 0, \quad \alpha\beta = 0.$$

Now $\alpha = 0$ is inadmissible; for then β would be equal γ while these magnitudes have opposite signs. The only possibility is, therefore,

$$\beta = 0, \quad \alpha = \gamma,$$

i.e. $\qquad n_2 = 0, \quad n_1^2(v_2^2 - v_3^2) = n_3^2(v_1^2 - v_2^2).$

But $\mathbf{n}^2 = 1$, so that $n_3^2 = 1 - n_1^2$. Ultimately, therefore, the required directions of the normal are

$$n_1 = \pm\sqrt{\frac{v_1^2 - v_2^2}{v_1^2 - v_3^2}}, \quad n_2 = 0, \quad n_3 = \pm\sqrt{\frac{v_2^2 - v_3^2}{v_1^2 - v_3^2}} \qquad (38)$$

These particular directions are called *the optical axes* of the crystal. One axis is obtained by taking in (38) the signs $+ +$ (or $- -$), the other axis by taking $+ -$ (or $- +$).

Thus, in the most general case, for different K_1, K_2, K_3, and therefore for different v_1, v_2, v_3, the crystal has *two optical axes* contained in the plane $\mathbf{i}$, $\mathbf{k}$, that is to say, in the plane of those two electrical axes to which correspond the

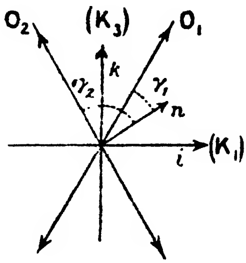

FIG. 2.

smallest and the greatest principal values of the permittivity-operator K. The orientation of the optical axes is symmetrical with respect to the electrical axes $\mathbf{i}$ and $\mathbf{k}$.

Substituting (38) in (37), we have

$$v' = v'' = v_2 \qquad (39)$$

This then is the common value of the two velocities when the wave-normal $\mathbf{n}$ coincides with one of the optical axes. It is easy to show that in this case the light-vector $\mathbf{D}$ can have *any* direction in the wave-plane. Independently of the orientation

of D the velocity of the wave is, in these circumstances, always equal v_2.

If γ_1, γ_2 be the angles which any wave-normal **n** makes with the two optical axes A_1, A_2 (as in Fig. 2), formulæ (37) for the corresponding velocities of propagation can be written

$$\begin{matrix} v' \\ v'' \end{matrix} = \frac{1}{\sqrt{2}} \cdot \sqrt{v_1^2 + v_3^2 + (v_1^2 - v_3^2) \cdot \cos(\gamma_1 \mp \gamma_2)} \qquad (40)$$

10. Uniaxial Crystals.

If the electrical properties of the crystal are axially symmetric, say, with respect to the axis **k**, in other words, if $K_2 = K_1$, and therefore

$$v_2 = v_1,$$

then (38) becomes

$$n_1 = 0, \qquad n_2 = 0, \qquad n_3 = \pm 1.$$

That is to say, the two optical axes coincide with the axis of electrical symmetry to which corresponds the principal permittivity K_3, *i.e.* the velocity v_3. The crystal is then *uniaxial.*

Since, in the present case, $\gamma_1 = \gamma_2 = \gamma$, say, so that $\cos\gamma = n_3$, the two velocities of propagation corresponding to any given direction of wave-normal **n**, making with the optical axis the angle γ, are, by (40),

$$\left.\begin{aligned} v' &= v_1 \\ v'' &= \sqrt{v_1^2 \cos^2\gamma + v_3^2 \sin^2\gamma} \end{aligned}\right\} \quad . \quad . \quad (41)$$

The velocity v', which is constant, corresponds to what is called the *ordinary*, and v'' depending upon the angle γ corresponds to the *extraordinary* wave. To these waves in which the light-vector **D** has one of the orthogonal directions

s′, **s**″ correspond, in general, two different rays, the *ordinary* ray **p**′, and the *extraordinary* ray **p**″.

A discussion of further details would not answer the purposes of this little volume. The reader will find them in every work on optics, whether based upon the electromagnetic or upon the older theory.

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